A Combinatorial Analysis of Subcube Reliability in Hypercubes

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***Abstract*—In this brief contribution, we derive an exact expression for $(n-1)$-cube reliability in an n-cube using a new probability fault model and an existing random fault model. Approximate results are also obtained for m-cube reliability for values of m smaller than $n-1$. We show that the proposed probability model for computing subcube reliability is equally accurate, but computationally more efficient than the existing random fault model.**

***Index Terms*—Hypercube, subcube reliability, fault-tolerance, combinatorial model.**

I. INTRODUCTION

Hypercube structures have received much attention due attractive properties of their topology such as logarithmic diameter, regularity, fault tolerance, and embeddability. Many commercial hypercube multiprocessors are available [1], [2], [3]. As the size of a system grows, the probability of a fault occurring in the system increases. It is important to quantify the effect of the faults, so that fault-tolerant designs can be pursued.

Reliability and availability are two measures normally used to evaluate the fault tolerance of a multiprocessor. The *reliability* of a system as a function of time, $R(t)$, is defined as the probability that the system has survived the interval $[0, t]$, given that it was operational at time $t = 0$. The *availability* of a system as a function of time, $A(t)$, is the probability that the system is operational at the instant of time t. If the limit of this function exists as t goes to infinity, availability expresses the expected fraction of time that the system is operational. Notice that the availability of a system is improved not only by the reliability of the components of the system but also by the maintenance and repair capability of the system. Reliability is used to describe systems in which repair cannot take place, such as satellite and aircraft computers.

A traditional measure of reliability evaluation is the *terminal reliability* of a computer network [4], [5]. Others are *task-based reliability* [6], [7] defined as the probability that some minimum number of connected nodes are available in the system for task execution, and *subcube reliability* [8], defined as the probability that a subcube of a specified size is available in the system. Among these, the subcube reliability measure is the most practical because a user in the current hypercube multiprocessors is given a specific subcube for the execution of his/her program.

In this brief contribution, we define two combinatorial reliability models, the *probability fault model* and the *random fault model*, to evaluate subcube reliability of hypercubes. In the probability fault model, the subcube reliability is directly computed by using the node reliability. In the random fault model, as shown in [7], [8], an indirect approach is used. The probability that there exist f faults in an n-cube is first calculated. Then the probability that there exists fault-free subcube in an n-cube with f faults is computed. By combining these two probabilities, subcube reliability is obtained. Approximate results for m-cube reliability are obtained in [7] using the random fault model. The same model is used in [8] to obtain $(n-1)$-cube reliability

We derive an exact expression for $(n-1)$-cube reliability and show the equivalence between two models. The $(n-1)$-cube reliability is obtained by forming all the available $(n-1)$-cubes first, and then applying the principle of inclusion and exclusion [9] to get the exact expression. We also employ a divide-and-conquer approach to analyze the subcube reliability of smaller sizes by using the probability fault model. Finally, we show that the probability fault model is computationally more efficient than the random fault model for calculating subcube reliability of smaller sizes.

We assume that processors in the system have a homogeneous reliability function. The failures of processors are assumed to be statistically independent and link failures are negligible compared to processor failures. The reliability function of each node can have any failure distribution. The node reliability function can include the maintenance and repair capabilities of the node.

The rest of this brief contribution is organized as follows. Section II presents some definitions and notations. In Section III, the exact symbolic expression for $(n-1)$-cube reliability is obtained by both the models. In Section IV, a divide-and-conquer technique is used to derive the subcube reliability of size $n-2$ or less. Numerical results are given in Section V, and the concluding remarks are provided in the last section.

II. NOTATIONS AND DEFINITIONS

An n-dimensional hypercube, denoted by Q_n, consists of $N = 2^n$ nodes. Each of the N nodes is addressed by a distinct binary string of length n. Two nodes are connected by a link if and only if their addresses differ in exactly one bit. Each subcube can be uniquely represented as a ternary string over the set {0, 1, *}, called its address, where * is a *Don't Care* symbol. Specifically, a d-dimensional subcube, called *d-cube*, has exactly d * s in its address, as it involves a group of 2^d nodes. For example, 00**, or equivalently 0^2*^2, represents the 2-cube formed by nodes 0000, 0010, 0001, and 0011 in a 4-cube. The *intersection* of two subcubes is the set of processors that is common to both the subcubes.

A. Notation

N: $N = 2^n$, the number of nodes in an n-cube.

$Q_{n-1}(a_i)$: $Q_{n-1}(a_i)$ is defined as $*\ldots*a_i*\ldots*$ which is an $(n-1)$-cube in an n-cube such that the ith bits of its ternary address is a_i(0 or 1) and all other bits are * s. For example, the 2-cube **1 in a 3-cube can be represented as $Q_2(a_0)$ where $a_0 = 1$.

$R_{n-1}(a_i)$: the reliability of the $(n-1)$-cube $Q_{n-1}(a_i)$.

$Q_{n-d}(a_{i_1}\ldots a_{i_d})$: $Q_{n-d}(a_{i_1}\ldots a_{i_d})$ is defined as an $(n-d)$-cube in an n-cube such that the i_jth bit of its ternary address is a_{i_j} (0 or 1) for $j = 1..d$ and all other bits are * s. For example, the 1-cube 0*1 in a 3-cube can be represented as $Q_2(a_2a_0)$ where $a_2 = 0$ and $a_0 = 1$.

complementary pair: $Q_{n-d}(a_{i_1}\ldots a_{i_j}\ldots a_{i_d})$ and $Q_{n-d}(a_{i_1}\ldots \overline{a_{i_j}}\ldots a_{i_d})$ are called a complementary $(n-d)$-cube pair in an n-cube.

p: p is the node reliability which is defined as the probability that the node is operational at time t.

$R_{n,m}(p)$: the m-cube reliability which is defined as.the probability that there exists a fault-free m-cube in an n-cube, given that p is the reliability of each node in the n-cube.

$P_n(f)$: the probability that there are f faulty nodes in an n-cube.

$P_{n,m}(f)$: the probability that there exists a fault-free m-cube in an n-cube, given there are f faulty nodes in the n-cube.

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C^n_m: $C^n_m = \frac{n\cdot(n-1)\cdots(n-m+1)}{m!}$ which is the number of combinations of m components selected from a set of n components.

S^n_m: $C^n_m \times 2^{n-m}$, i.e., the number of m-cubes in an n-cube.

$\lceil x \rceil$: the smallest integer that is greater than or equal to x.

$\lfloor x \rfloor$: the largest integer that is smaller than or equal to x.

III. ANALYSIS OF $R_{n,n-1}(p)$ AND $P_{n,n-1}(f)$

In Section III we calculate the exact expressions for $R_{n,n-1}(p)$ and $P_{n,n-1}(f)$ which will be used later to derive $R_{n,m}(p)$ and $P_{n,m}(f)$ for $1 \le m \le n-2$. We prove that the $(n-1)$-cube reliability are equivalent with both the models.

In the probability fault model, the reliability of each node at time t is a random variable. The probability that a subcube is operational is represented by the reliability of the processors in the subcube. The m-cube reliability of a hypercube can be formulated as the union of the probabilistic events that all the possible m-cubes are operational. Since the terms in the m-cube reliability formula obtained above may not be mutually disjoint, a technique to convert the reliability formula into one with only mutually disjoint terms is needed. The basic method to compute the network reliability used in the probability fault model is called the *principle of inclusion and exclusion* [9]. The principle of inclusion and exclusion is not efficient for calculating the reliability of general networks. However, we show that it is very useful for hypercube reliability analysis.

In the random fault model, we fix f, the number of faulty nodes in an n-cube, as $0 \le f \le N$. The rationale behind fixing f faulty nodes in the n-cube is that we can easily obtain the probability $P_n(f) = C^N_f p^{N-f}(1-p)^f$ that there are f faults in an n-cube. Thus, the remaining task is to compute the probability $P_{n,m}(f)$.

The primary task of computing $P_{n,m}(f)$ is to count the number of distributions of f faults out of C^N_f distributions such that there exists a fault-free m-cube. Then the system reliability can be expressed as

$$R_{n,m}(p) = \sum_{f=0}^{N} P_n(f) \times P_{n,m}(f). \qquad (1)$$

A. The Probability Fault Model

In this model, all the $2n$ $(n-1)$-cubes are formed first. Then the $(n-1)$-cube reliability is expressed as the union of the reliability of these $2n$ $(n-1)$-cubes. Since the probabilistic terms in the expression of the $(n-1)$-cube reliability are not mutually disjoint, the key to calculate the $(n-1)$-cube reliability is to convert the original reliability expression into one containing only disjoint terms. Let C_i be the probability that an $(n-1)$-cube is operational, the $(n-1)$-cube reliability can be represented by the principle of inclusion and exclusion [9] as follows.

$$\begin{aligned} R_{n,n-1}(p) = &\sum_{i=0}^{2n-1} C_i + (-1)\sum_{i,j=0\ldots 2n-1}^{i\ne j} C_iC_j(\text{all pairs}) \\ &+ (-1)^2 \sum_{i,j,k=0\ldots 2n-1}^{i\ne j\ne k} C_iC_jC_k(\text{all triples}) + \ldots \qquad (2) \\ &+ (-1)^{2n}\prod_{i=0}^{2n-1} C_i. \end{aligned}$$

Each C_i can be represented by the reliability of the 2^{n-1} processors in its corresponding $(n-1)$-cube. In the following, we derive a generalized result for $R_{n,n-1}(p)$ of an n-cube. We consider a simple case first. The 0-cube reliability can be easily obtained as $R_{n,0}(p) = 1-(1-p)^N$ because the only instance where there is no fault-free 0-cube in an n-cube is when all the nodes are faulty. Before deriving the main result, we need the following lemmas.

LEMMA 1. *There are S^n_{n-d} different combinations of d $(n-1)$-cubes in an n-cube such that the intersection of these d $(n-1)$-cubes is an $(n-d)$-cube, for $2 \le d \le n$.*

PROOF. There are C^{2n}_d combinations of d $(n-1)$-cubes out of $2n$ $(n-1)$-cubes. However, if the set of d $(n-1)$-cubes contain a complementary $(n-1)$-cube pair, the common intersection of these d $(n-1)$-cubes is empty because $Q_{n-1}(a_i)$ and $Q_{n-1}(\overline{a_i})$ are mutually disjoint. Any set of d $(n-1)$-cubes that contains no complementary $(n-1)$–cube pair intersects in a common $(n-d)$-cube. For example, the intersection of d $(n-1)$-cubes, $Q_{n-1}(a_{i_1})$, $Q_{n-1}(a_{i_2})$, ..., and $Q_{n-1}(a_{i_d})$ for $a_{i_j} = 0$ or 1, $i_j \ne i_k$, $j \ne k$, and $1 \le j, k \le d$, is the $(n-d)$–*cube*, $Q_{n-d}(a_{i_1}, a_{i_2}, \ldots, a_{i_d})$. The number of such sets of d $(n-1)$-cubes is $C^n_d \times 2^d = S^n_{n-d}$ where d $(n-1)$-cubes are selected from n complementary pairs and there are two choices in each selected complementary $(n-1)$-cube pair. □

LEMMA 2. *The total number of nodes in the d $(n-1)$-cubes which intersect in an $(n-d)$-cube is equal to $2^n - 2^{n} - d$.*

PROOF. We shall see that any k of these d $(n-1)$-cubes intersect in an $(n-k)$-cube. Thus, by the principle of inclusion and exclusion, the number of nodes in these d $(n-1)$-cubes can be obtained as $2^{n-d}\sum_{i=1}^{d}(-1)^{i-1}C^d_i 2^{d-i} = 2^n - 2^{n-d}$, i.e., the sum of the number of nodes in these d $(n-1)$-cubes, subtracted by the number of nodes in the intersections of any two $(n-1)$-cubes, added by the number of nodes in the intersections of any three $(n-1)$-cubes, and so on. □

For example, the intersection of two 2-cubes 0** and **0 is 0*0. It is easy to see that there are S^3_1 different pairs of two 2-cubes intersecting in a 1-cube. The intersection of 0**, **0, and 0*0 is 000.

THEOREM 2. *Given a homogeneous node reliability p in an n-cube, the $(n-1)$-cube reliability is*

$$R_{n,n-1}(p) = \sum_{i=1}^{n}(-1)^{i-1} S^n_{n-i} p^{N-2^{n-i}} + (-1)^n p^N \qquad (3)$$

PROOF. According to the principle of inclusion and exclusion, the probability of having a fault-free $(n-1)$-cube is obtained as

$$\begin{aligned} R_{n,n-1}(p) = &\sum_{i=0}^{2n-1} R_{n-1}(a_i) + (-1)\sum_{i,j=0\ldots 2n-1}^{i\ne j} R_{n-1}(a_i)\,R_{n-1}(a_j) \\ &+ (-1)^2 \sum_{i,j,k=0\ldots 2n-1}^{i\ne j\ne k} R_{n-1}(a_i)\,R_{n-1}(a_j)\,R_{n-1}(a_k) + \ldots \qquad (4) \\ &+ (-1)^{2n-1}\prod_{i=0}^{2n-1} R_{n-1}(a_i), \end{aligned}$$

where $a_i = 0$ for $0 \le i \le n-1$ and $a_i = 1$ for $n \le i \le 2n-1$. A set of $(n-1)$-cubes covers all the N nodes if the set contains a complementary $(n-1)$-cube pair. Otherwise, as shown in Lemma 2, a set of $(n-1)$-cubes containing no complementary $(n-1)$-cube pair covers fewer than N nodes. Thus, the terms in (4) can be divided into two cases: one is for the terms covering fewer than N nodes and the other is for the terms covering all the N nodes.

Case I. We first consider the terms in (4) that do not cover all the N nodes in an n-cube, e.g., $(-1)\ Q_{n-1}(a_i) \times Q_{n-1}(a_j)$, for $i \neq j$ mod n or $j \neq i$ mod n. According to Lemmas 1 and 2, the terms $((-1)^{i-1} S^n_{n-i} p^{N-2^{n-i}})$ in the summation form of (3) can be obtained. For example, any two $(n-1)$-cubes, excluding the complementary $(n-1)$-cube pairs, intersect in an $(n-2)$-cube. The number of nodes covered by these two $(n-1)$-cubes is $2^n - 2^{n-2}$ according to Lemma 2. Since there are $C^n_2 \times 2^2$ pairs which are not complementary $(n-1)$-cube pairs, we get the term $(-1)S^n_{n-2}\ p^{N-2^{n-2}}$. Other terms can be derived similarly.

Case II. Consider the terms consisting of k $(n-1)$-cubes, $2 \leq k \leq 2n$. The terms consisting of a complementary $(n-1)$-cube pair must belong to this case. Again by using the principle of inclusion and exclusion, we have B_k choices such that these terms cover all the nodes in the n-cube, as follows.

$$\begin{aligned} B_k = p^N \times \Big[& C^n_i C^{2n-2}_{k-2} + (-1)^1 C^n_2 C^{2n-4}_{k-4} + \dots \\ & + (-1)^{i-1} C^n_i C^{2n-2i}_{k-2i} + \dots \\ & + (-1)^{\lfloor \frac{k}{2} \rfloor - 1} C^n_{\lfloor \frac{k}{2} \rfloor} C^{2n-2\lfloor \frac{k}{2} \rfloor}_{k-2\lfloor \frac{k}{2} \rfloor} \Big] \end{aligned} \tag{5}$$

The first item $C^n_1 C^{2n-2}_{k-2}$ in the square brackets includes all the terms containing at least one complementary $(n-1)$-cube pair. However, the terms containing at least two complementary pairs are included twice in B_k. Therefore, $C^n_2 C^{2n-4}_{k-4}$ is subtracted from B_k. Similarly, we add the terms containing at least three complementary pairs, and so on, until $\lfloor \frac{k}{2} \rfloor$ complementary pairs are included. Summing up all B_k, we have the reliability for the second case as

$$B = p^N \times \sum_{k=2}^{2n} (-1)^{k-1} B_k = (-1)^n p^N \tag{6}$$

Adding the reliability of the above two cases, we obtain the proof for the theorem. □

B. The Random Fault Model

An approximate evaluation of $P_{n,m}(f)$, using the random fault model, exists in the literature [7]. In this section, we only consider $P_{n,n-1}(f)$ and show that an exact expression can be achieved for this particular case. Then we show that we obtain the same results as those using the probability fault model.

We start with a simple example involving a 3-cube with two faults. To have a fault-free 2-cube, these two faults must reside in a 2-cube such that the other 2-cube is fault-free. In total, there are $S^3_2 = 6$ 2-cubes in a 3-cube. There are C^4_2 different distributions of two faults in a 2-cube. Therefore, there are $6 \times C^4_2 = 36$ possible configurations where the two faults are located in a 2-cube. However, many cases are counted twice. For example, if the faults are located at nodes 0 and 1, they are counted twice for both 0** and *0*. Thus, the actual number of distributions of these two faulty nodes in a 2-cube is $S^3_2 \times C^4_2 - S^3_1 \times C^2_2 = 24$. The result of above example can be verified by an independent argument as follows. Given two faults in an n-cube, the only distribution of the two faults which causes no $(n-1)$-cube to be available is the one where two faults are located at antipodal positions. Thus, the above result can be easily verified as $C^8_2 - 4 = 24$. Before we provide the general result for the n-cube containing any number of faults, we need the following lemma.

LEMMA 3. *The number of distributions of f faults which are confined in an $(n-d)$-cube is $N_{n,n-d}(f)$, for $1 \leq d \leq n$, where*

$$N_{n,n-d}(f) = \sum_{i=d}^{n-\lceil \log f \rceil} (-1)^{i-d} S^n_{n-i} C^{2^{n-i}}_f . \tag{7}$$

PROOF. This proof follows from the principle of inclusion and exclusion. There are $S^n_{n-d} C^{2^{n-d}}_f$ different ways in which f faults can be distributed in an $(n-d)$-cube. However, there is a possibility that the faults distributed in an $(n-d-1)$-cube are counted twice. Therefore, $S^n_{n-d-1}\ C^{2^{n-d-1}}_f$ needs to be deducted from the sum to rectify the situation. Again, the possibility that faults are contained in an $(n-d-2)$-cube needs to be added back, and so on. □

THEOREM 3. *The $(n-1)$-cube reliability in an n-cube containing $f > 1$ faulty nodes is*

$$P_{n,n-1}(f) = \frac{N_{n,n-1}(f)}{C^N_f} \tag{8}$$

PROOF. It directly follows from Lemma 3. □

The above results can be easily verified when $f = 1$ or 2, as follows. The number of distributions of one fault in an n-cube is $N_{n,n-1}(1) = \sum_{i=1}^{n} (-1)^{i-1} S^n_{n-1} C^{2^{n-i}}_1 = N$. The number of distributions of two faults in an n-cube is $C^{2^n}_2 - 2^{n-1} = N_{n,n-1}(2)$ because the distributions with two faults in antipodal positions are the only ones that destroy all the $(n-1)$-cubes.

THEOREM 4. *The $(n-1)$-cube reliability in a faulty n-cube derived by the probability fault model and the random fault model are equivalent.*

PROOF. Given f faults in an n-cube, Theorem 3 gives the probability of having a fault-free $(n-1)$-cube as $\frac{N_{n,n-1}(f)}{C^N_f}$. With a node reliability p, the probability of having f faults in the n-cube is $C^N_f p^{N-f} (1-p)^f$. Thus, the random fault model gives the $(n-1)$-cube reliability as

$$\begin{aligned} & p^N + \sum_{f=1}^{N} \sum_{i=1}^{n} (-1)^{i-1} S^n_{n-i}\ C^{2n-i}_f \times p^{N-f} \times (1-p)^f \\ & = (-1)^n p^N + \sum_{i=1}^{n} (-1)^{i-1} S^n_{n-1} \times p^{N-2^{n-i}} \\ & = R_{n,n-1}(p) \end{aligned}$$

in (3). The summation index f in the first line starts from 1 because the probability of all the nodes being good (i.e., p^N) is separated out in the first line. □

IV. APPROXIMATE ANALYSIS OF $R_{n,m}(p)$

In Section IV, we derive an approximate subcube reliability expression for $1 \leq m \leq n-2$. Since an analysis of $P_{n,m}(f)$ using the random fault model exists in the literature [7], [10], we only present the derivation of $R_{n,m}(p)$ using the proposed probability fault model. It is shown in Section V that the new model is as accurate as the random fault model, but is more efficient computationally.

We use a divide-and-conquer technique to compute $R_{n,m}(p)$. The n-cube is divided into two $(n-1)$-cubes, Q_{n-1} and Q'_{n-1}, along dimension i. We consider two cases in which a fault-free $(n-1)$-cube can be formed as follows.

1) There exists a fault-free m-cube in either Q_{n-1} or Q'_{n-1}.
2) Neither Q_{n-1} nor Q'_{n-1} contain a fault-free m-cube. But a fault-free m-cube can be formed by combining a fault-free $(m-1)$-cube in Q_{n-1} with its complementary $(m-1)$-cube in Q'_{n-1}.

Subsequently, we derive the probabilities of having a fault-free m-cube in an n-cube, $R^1_{n,m}(p)$ and $R^2_{n,m}(p)$ for Cases I and II, respectively.

Case I. Since we know the probability of having a fault-free $(n-1)$-cube in an n-cube, the probability of having a fault-free m-cube in either Q_{n-1} or Q'_{n-1} may be written as $R_{n-1,m}(p)$. Thus, the probability $R'_{n,m}$ of having an m-cube in an n-cube can be expressed as

$$R^1_{n,m}(p) = 2 \times R_{n-1,m}(p) - [R_{n-1,m}(p)]^2. \tag{9}$$

The last negative term within square brackets denotes the case when both Q_{n-1} or Q'_{n-1} have a fault-free m-cube. This term is subtracted because it is the probability that is counted twice.

Case II. The only way that an $(m-1)$-cube in Q_{n-1} and an $(m-1)$-cube in Q'_{n-1} can be combined to form an m-cube is that these two $(m-1)$-cubes must be neighbors. In other words, the ternary representations of these two $(m-1)$-cubes are different only in the ith bit, making them a complementary $(m-1)$-cube pair. Thus, we can interpret the n-cube as an $(n-1)$-cube in which each node is a one-cube. Thus, the problem of determining the probability of having a fault-free m-cube for Case II reduces to determining the probability of having a fault-free $(m-1)$-cube in an $(n-1)$-cube with a node reliability p^2. Therefore, we have the following probability $R^2_{n,m}(p)$ for Case II:

$$R^2_{n,m}(p) = R_{n-1,m-1}(p^2) - R_{n-1,m}(p^2). \tag{10}$$

In (10) the deduction of $R_{n-1,m}(p^2)$ from $R_{n-1,m-1}(p^2)$ avoids counting twice the probability of having m-cubes in each of the $(n-1)$-cubes.

The probability expressions of $R^1_{n,m}(p)$ and $R^2_{n,m}(p)$ are not disjoint. We cannot simply sum up $R^1_{n,m}(p)$ and $R^2_{n,m}(p)$ and then deduct $R^1_{n,m}(p) \times R^2_{n,m}(p)$ because the value of $R^1_{n,m}(p) \times R^2_{n,m}(p)$ is smaller than what is supposed to be subtracted. The reason is as follows. The value of $R^1_{n,m}(p) \times R^2_{n,m}(p)$ is obtained as if the random variables in $R^1_{n,m}(p)$ were different from those in $R^1_{n,m}(p)$. For example, consider the probability of having a fault-free 1-cube in a 3-cube. The reliability expression of $R^1_{3,1}(p)$ includes x_0x_1, where x_i denotes the reliability of node i in the 3-cube. The reliability expression of $R^2_{3,1}(p)$ includes x_0x_4. The union of x_0x_1 and x_0x_4 is $x_0x_1x_4$ which should be subtracted. As can be seen, the reliability expression of $R^1_{3,1}(p) \times R^2_{3,1}(p)$ includes $x_0x_1x_0x_4$ instead of $x_0x_1x_4$. In other words, some information about the probability contribution of $R^1_{3,1}(p)$ and $R^2_{3,1}(p)$ is lost. However, it is difficult to obtain the exact expression for the m-cube reliability. Therefore, for simplicity, we approximate the probability of having an m-cube as follows.

$$R_{n,m}(p) = R^1_{n,m}(p) + R^2_{n,m}(p) - \frac{R^1_{n,m}(p)}{R_{n-1,m-1}(p)} \times \frac{R^2_{n,m}(p)}{R_{n-1,m-1}(p)} \tag{11}$$

Equation (11) is a recursive expression since $R^2_{n,m}(p)$ is recursive. The recursion continues until certain boundary conditions, $R_{n,n-1}(p)$ and $R_{2,0}(p)$, are reached.

V. Numerical Results

In Section V, we plot and compare numerical results for subcube reliability of different sizes. The results in [8] were obtained for the $(n-1)$-cube and were later compared with the results in [7]. Hence, we will compare our results only with those obtained in [7] using the random fault model.

Fig. 1 illustrates the subcube reliability for a 6-cube. The node reliability is assumed to be homogeneous with an exponential failure distribution rate $\lambda = 0.001$. The 5-cube reliability in a 6-cube is calculated by (3) which is exact and denoted as *Exact*65 in the figure. The number 65 refers to the reliability of 5-cube in a 6-cube system. The results, *Random*65 and *Prob*65, obtained with the random and the probability fault models are the same as *Exact*65 and, therefore, are not shown in Fig. 1. *Exact*64 (4-cube reliability in a 6-cube) is the exact result computed by using the Boolean algebra technique which converts the reliability expression into one with only disjoint terms. *Prob*64 and *Random*64 (*Prob*63 and *Random*63) represent the 4-cube (3-cube) results of the probability fault model and the random fault model, respectively. *Exact*63 is not shown because it can not be computed in a reasonable amount of time by using the Boolean algebra technique. As expected, the reliability of a smaller subcube is higher than that of a larger subcube. It is seen that the difference among *Exact*64, *Prob*64, and *Random*64 is not significant. Fig. 2 illustrates similar reliability results for an 8-cube based on $\lambda = 0.0001$.

The time complexity of calculating $R_{n,m}(p)$ for the probability fault model can be analyzed as follows. According to (11), $R_{n,m}(p)$ is obtained by adding up all the $R_{i,j}(p)$s for $i = n-1$ to 0 and $j = m$ to 0. Assuming that the time to calculate each item in $R_{n,n-1}(p)$ is $O(1)$, the time complexity of $R_{n,n-1}(p)$ is $O(n)$. Thus, the time complexity of $R_{n,m}(p)$ can be obtained as $O(n^2)$. The time complexity of calculating the m-cube reliability using the random fault model is $O(n^2 2^n)$, the detailed derivation being given in [10]. The random fault model has much higher complexity than the probability fault model because it involves the computations of the reliability for number of faults from 1 to 2^n.

The low time complexity of the new probability fault model is evident from the following experiment. Table I shows the execution time (in seconds) of computing the subcube reliability using the probability and random fault models, as denoted by ProbFM and RandFM, respectively. The computation time for the random fault model is obtained by using the recursive reliability formulae given in [7]. The measurement of time is based on a C program running on a SPARC Station (model IPC). The 0 second shown in the table for the probability fault model means the computation is too fast to be measured in seconds. Finally, we show the subcube reliability for a large hypercube of size 12 in Fig. 3 with $\lambda = 0.0001$. It only shows the results from the probability fault model since the random fault model is too time consuming to compute the reliability.

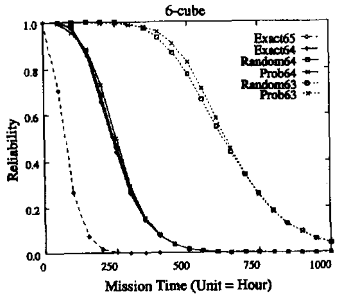

Fig. 1. Subcube reliability variation of a 6-cube.

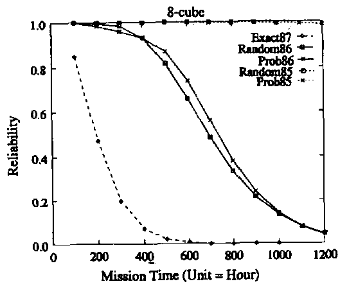

Fig. 2. Subcube reliability variation of an 8-cube.

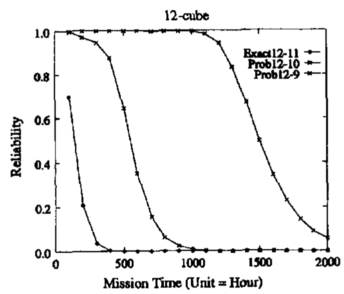

Fig. 3. Subcube reliability variation of a 12-cube.

TABLE I
COMPUTATION TIME (SEC) OF SUBCUBE RELIABILITY ON A SPARC STATION

	ProbFM	RandFM
6-cube	0	4
7-cube	0	54
8-cube	0	751
9-cube	0	11,700

VI. CONCLUSION

We have presented two fault models, the probability fault model and the random fault model, to predict the subcube reliability of hypercubes. An exact expression for $(n-1)$-cube reliability is derived for both the models. The divide-and-conquer technique is used to obtain the reliability of subcubes of smaller sizes. A technique to obtain approximate results for the subcube reliability of size smaller than $n-1$ is presented using the probability fault model. We showed that the new probability fault model is much more computationally efficient than the random fault model for calculating subcube reliability of smaller sizes.

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